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Note

A local independence number condition for n-extendable graphs

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Abstract

Let G be a graph and $v \in V(G)$. Let $N_k(v) = \{u \mid u \in V(G)$ and $d(u, v) = k\}$. It is proved that if G is a connected graph with $\infty > g(G) \geqslant 4$ and with even order and if, for each vertex v in $V(G)$, $\alpha(G[N_2(v)]) \leqslant d(v) - 1$, then G is regular and $\lceil d(v)/4 \rceil$-extendable. All results in this paper are sharp. © 1999 Elsevier Science B.V. All rights reserved

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All graphs considered are finite, undirected and simple.

Let n be a positive integer and G be a graph with $v \geqslant 2n + 2$. G is said to be n-extendable if G has n independent edges and any n independent edges of G are contained in a perfect matching of G.

Let G be a graph and $v \in V(G)$. Let $N_k(v) = \{u \mid u \in V(G)$ and $d(u, v) = k\}$. Let $S \subseteq V(G)$ and H be a subgraph of G. Then we use the notation $N_S(v) = N(v) \cap S$ and $N_H(v) = N(v) \cap V(H)$. And $d_S(v) = |N_S(v)|$ and $d_H(v) = |N_H(v)|$. Let x be a real number. We denote by $\lceil x \rceil$ the least integer m such that $m \geqslant x$ and by $\lfloor x \rfloor$ the largest integer m such that $m \leqslant x$.

For the terminology and notation not defined in this paper, the reader is referred to [3].

The concept of n-extendable graphs was introduced by Plummer [7]. Since then, a lot of work on this topic has been done (for example, see [1,4,6–9]). For the advances in this topic, the reader could be referred to [9].

Plummer [7,8] and Lou [4,6] gave some sufficient conditions for n-extendable graphs. Aldred et al. [1] showed that symmetric graphs with large girth are n-extendable. In the light of this result, we give a sufficient condition for n-extendable graphs which

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requires some sort of symmetry and girth at least 4. Lou [5] proved that under the above condition a graph is also hamiltonian.

In the following, we shall give the main results of this paper.

Theorem 1. *Let G be a connected graph with $\infty > g(G) \geqslant 4$ and with even order. If, for each vertex v in $V(G), \alpha(G[N_2(v)]) \leqslant d(v) - 1$, then G is regular and G is $\lceil d(v)/4 \rceil$-extendable.*

Proof. Let G be a graph satisfying the hypotheses of this theorem. First, we have two claims.

Claim 1. *G is regular.*

Suppose G is not regular. Then there are two adjacent vertices u and v in G with $d(u) > d(v)$. Since $g \geqslant 4$, $N(u) \backslash \{v\} \subseteq N_2(v)$ and $N(u)\backslash\{v\}$ is an independent set with $|N(u)\backslash\{v\}| \geqslant d(v)$, contradicting the hypothesis. The proof of Claim 1 is complete.

Claim 2. *The diameter of G is 2.*

Suppose, to the contrary, there are two vertices u and v in G such that $d(u,v) = 3$. Let $uwxv$ be a path from u to v in G. Then $(N(u)\backslash\{w\}) \cup \{v\} \subseteq N_2(w)$ is an independent set as $g \geqslant 4$ and $d(u,v) = 3$. And $|(N(u)\backslash\{w\}) \cup \{v\}| = d(u) - 1 + 1 = d(u) = d(w)$ by Claim 1, contradicting the hypothesis. The proof of Claim 2 is complete.

Let $d(v) = k$ for all $v \in V(G)$. Suppose G is not $\lceil k/4 \rceil$-extendable. Then there are $\lceil k/4 \rceil$ independent edges $e_i = u_i v_i$ $(i = 1, 2, \ldots, \lceil k/4 \rceil)$ such that $G' = G - \{u_i, v_i \mid i = 1, 2, \ldots, \lceil k/4 \rceil\}$ has no perfect matching. By Tutte's Theorem, there is a set $S' \subseteq V(G')$ such that $o(G' - S') > |S'|$. By parity, $o(G' - S') \geqslant |S'| + 2$. Let $S = S' \cup \{u_i, v_i \mid i = 1, 2, \ldots, \lceil k/4 \rceil\}$. Then $o(G - S) = o(G' - S') \geqslant |S'| + 2 = |S| - 2\lceil k/4 \rceil + 2$. So we have

$$|S| - o(G - S) \leqslant 2\lceil k/4 \rceil - 2. \tag{1}$$

Claim 3. *$G - S$ has at most k components.*

Let x be a vertex in a component of $G - S$. By Claim 2, the vertices of the other components are in $N_2(x)$. By the hypothesis of this theorem, Claim 3 follows as any two vertices in two different components are independent. Now we have two cases to study.

Case 1. $G - S$ has an even component C.

Case (1.1): $|V(C)| = 2$.

Assume $G - S$ has t odd components $C_1, C_2, \ldots, C_t$. Let α_i be the independence number of C_i $(i = 1, 2, \ldots, t)$. Without loss of generality, assume $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_t$. Let uv be the only edge in C. Since $g \geqslant 4$, $|S| \geqslant |(N(u) \cup N(v))\backslash\{u, v\}| = |N(u)\backslash\{v\}| + |N(v)\backslash\{u\}| = 2(k - 1)$.

Let M_i be a maximum independent set in C_i $(i=1,2,\ldots,t)$. By Claim 2, $M=\bigcup_{i=1}^{t} M_i$ is an independent set in $N_2(u)$. By the hypothesis, $|M|\leqslant k-1$. So

$$\begin{aligned}
&\mathrm{o}(G-S)\leqslant k-1-\sum_{i=1}^{t}(\alpha_i-1)=k+t-\sum_{i=1}^{t}\alpha_i-1,\\
&\quad |S|-\mathrm{o}(G-S)\\
&\qquad \geqslant 2(k-1)-\left[k+t-\sum_{i=1}^{t}\alpha_i-1\right]\\
&\qquad \geqslant 2(k-1)-(k+t)+t\alpha_1+1\\
&\qquad =k+t(\alpha_1-1)-1\\
&\qquad \geqslant k-1\\
&\qquad >2\lceil k/4\rceil-2
\end{aligned}$$

contradicting (1).

Case (1.2): $|V(C)|>2$.

Now $\alpha(C)\geqslant 2$. Assume that $G-S$ has t components such that each of their independence numbers is at least two. Let α_i $(i=1,2,\ldots,t)$ be their independence numbers. Without loss of generality, assume $\alpha_1\leqslant\alpha_2\leqslant\cdots\leqslant\alpha_t$. Let C_1 be the component with $\alpha(C_1)=\alpha_1$ and uv be an edge in C_1. Since $g\geqslant 4$, $N(u)\cap V(C_1)$ is an independent set in C_1. However, $\alpha(C_1)=\alpha_1$. So u sends at least $k-\alpha_1$ edges to S, so does v by the same reason. Since $g\geqslant 4$, $|S|\geqslant|N_S(u)|+|N_S(v)|\geqslant 2(k-\alpha_1)$. Let $t\geqslant 2$. By a similar reason as in Case (1.1)

$$\begin{aligned}
&\mathrm{o}(G-S)\leqslant k-\sum_{i=1}^{t}(\alpha_i-1)-1=k+t-\sum_{i=1}^{t}\alpha_i-1,\\
&\quad |S|-\mathrm{o}(G-S)\\
&\qquad \geqslant 2(k-\alpha_1)-\left[k+t-\sum_{i=1}^{t}\alpha_i-1\right]\\
&\qquad \geqslant k+(t-2)\alpha_1-(t-1)\\
&\qquad =k-1+(t-2)(\alpha_1-1)\\
&\qquad \geqslant k-1\\
&\qquad >2\lceil k/4\rceil-2,
\end{aligned}$$

contradicting (1).

Now let $t=1$. Then all odd components of $G-S$ are singletons and C is an even component with $|V(C)|>2$ in $G-S$. Let q be the independence number of C and m be the number of odd components in $G-S$. Let x be a singleton component in $G-S$. Since G is k-regular and $g\geqslant 4$, besides u_i and v_i $(i=1,2,\ldots,\lceil k/4\rceil)$, x is adjacent to at least $k-\lceil k/4\rceil$ vertices in S. So $|S'|\geqslant k-\lceil k/4\rceil$. By Claim 2 and the hypothesis, $q+k-\lceil k/4\rceil+2\leqslant q+m\leqslant k$. Hence, $q\leqslant\lceil k/4\rceil-2$. Now, let uv be an edge in C, both u and v send at least $k-q$ edges to S. Since $g\geqslant 4$, $|S|\geqslant 2(k-q)\geqslant 2(k-\lceil k/4\rceil+2)$.

So $|S'| = |S| - 2\lceil k/4 \rceil \geqslant 2(k - \lceil k/4 \rceil + 2) - 2\lceil k/4 \rceil \geqslant k$. Therefore, $m \geqslant |S'| + 2 \geqslant k+2$, contradicting Claim 3.

Case 2: $G-S$ has no even component.

Case (2.1): $G-S$ has $t \geqslant 1$ components with at least 3 vertices.

Let α_i $(i=1,2,\ldots,t)$ be the independence numbers of the odd components with order at least 3 in $G-S$. Without loss of generality, assume that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_t$ and C_1 is the component with $\alpha(C_1) = \alpha_1$. Then, we have the following claim.

Claim 4. $|S| \geqslant 2(k - \alpha_1) + 1$.

Since $g \geqslant 4$, $d_{C_1}(x) \leqslant \alpha_1$ for each vertex x of C_1. Suppose $uv \in E(C_1)$ such that $d_{C_1}(u) < d_{C_1}(v)$. $|S| \geqslant (k - d_{C_1}(u)) + (k - d_{C_1}(v)) \geqslant (k - \alpha_1 + 1) + (k - \alpha_1) = 2(k - \alpha_1) + 1$.

Now, we assume that C_1 is a regular graph and $d_{C_1}(v) = \alpha_1$ for all $v \in V(C_1)$. If $d_{C_1}(x) < \alpha_1$ for some $x \in V(C_1)$, the above argument can also apply. Let u and v be two adjacent vertices in C_1. If, for any $w \in N_{C_1}(u)$, $N_{C_1}(w) = N_{C_1}(v)$, then C_1 is $K_{m,m}$ and hence is not an odd component, a contradiction. So there is a vertex $w \in N_{C_1}(u)$ such that $N_{C_1}(w) \neq N_{C_1}(v)$. However, $N_{C_1}(v)$ is an independent set of order α_1 in C_1. Then every vertex in $N_{C_1}(w) \backslash N_{C_1}(v)$ sends an edge to $N_{C_1}(v)$. As G has no triangle, there is a 5-cycle $D = x_1x_2x_3x_4x_5x_1$ in C_1 such that $d_S(x_i) = k - \alpha_1$ $(i=1,2,3,4,5)$. Since $g \geqslant 4$ and then $N_S(x_1) \cap N_S(x_2) = \emptyset$, $|S| \geqslant 2(k - \alpha_1)$. Suppose, $|S| = 2(k - \alpha_1)$. Then $N_S(x_3) = N_S(x_1)$, $N_S(x_4) = N_S(x_2)$ and $N_S(x_5) = N_S(x_1)$. But $x_1x_5 \in E(G)$. Therefore, G has a triangle, contradicting the hypothesis that $g \geqslant 4$. Hence, $|S| \geqslant 2(k - \alpha_1) + 1$. Claim 4 holds.

Case (2.1.1): $t \geqslant 2$.

Now, we have $\mathrm{o}(G-S) \leqslant k - \sum_{i=1}^{t} (\alpha_i - 1)$,

$$
\begin{aligned}
&|S| - \mathrm{o}(G-S) \\
&\geqslant 2(k - \alpha_1) + 1 - \left[k - \sum_{i=1}^{t} (\alpha_i - 1) \right] \\
&\geqslant 2(k - \alpha_1) + 1 - k - t + t\alpha_1 \\
&= k + (t-2)(\alpha_1 - 1) - 1 \\
&\geqslant k - 1 \\
&> 2\lceil k/4 \rceil - 2
\end{aligned}
$$

contradicting (1).

Case (2.1.2): $G-S$ has exactly one odd component C with order at least 3.

Let q be the independence number of C and m be the number of odd components in $G-S$. Let x be a singleton component in $G-S$. Since G is k-regular and $g \geqslant 4$, besides u_i and v_i $(i=1,2,\ldots,\lceil k/4 \rceil)$, x is adjacent to at least $k - \lceil k/4 \rceil$ vertices in S. So $|S'| \geqslant k - \lceil k/4 \rceil$. By Claim 2 and the hypothesis, $q + k - \lceil k/4 \rceil + 1 \leqslant q - 1 + m \leqslant k$. Hence, $q \leqslant \lceil k/4 \rceil - 1$. By Claim 4, $|S| \geqslant 2(k-q) + 1$. So $|S'| = |S| - 2\lceil k/4 \rceil \geqslant 2(k - \lceil k/4 \rceil +$

$1)+1-2\lceil k/4\rceil = k+(k+3-4\lceil k/4\rceil)\geqslant k$. Hence, $m\geqslant |S'|+2\geqslant k+2$, contradicting Claim 3.

Case (2.2): All components of $G-S$ are singletons.

Let $C_1, C_2, \ldots, C_m$ be the components of $G-S$. First, we claim that there are C_i and C_j such that $N_S(C_i)\neq N_S(C_j)$ $(i\neq j)$. Suppose, to the contrary, that $N_S(C_1)=N_S(C_i)$ $(i=1,2,\ldots,m)$. Now, every vertex w in $S\backslash N_S(C_1)$ is adjacent to a vertex in $N_S(C_1)$ by Claim 2. Also, for each edge uv in $G[S]$, either u or v is in $N_S(C_1)$ but not both. If both u and v are in $N_S(C_1)$, then we have a contradiction to the hypothesis that $g\geqslant 4$. If neither u nor v is in $N_S(C_1)$, then $(N(u)\cup N(v))\backslash\{u,v\}\subseteq S$. Since $g\geqslant 4$, $|S|\geqslant |(N(u)\cup N(v))\backslash\{u,v\}|+|\{u,v\}| = |N(u)\backslash\{v\}|+|N(v)\backslash\{u\}|+|\{u,v\}| = 2k$. So $|S'|=|S|-2\lceil k/4\rceil\geqslant 2k-2\lceil k/4\rceil\geqslant k$ and $o(G-S)\geqslant |S'|+2>k$, contradicting Claim 3. Therefore, G is a bipartite graph with bipartition $(N_S(C_1), V(G)\backslash N_S(C_1))$. The only possible such graph is $K_{k,k}$. However, $K_{k,k}$ is $\lceil k/4\rceil$-extendable. So there is no such graph, a contradiction. The claim that $N_S(C_i)\neq N_S(C_j)$ for some i and j is proved.

Now, $|N_S(C_i)|=k$. Let w be a vertex in $N_S(C_i)\cap N_S(C_j)$. Then $N_S(C_i)\backslash\{w\}$ is an independent set of order $k-1$ in $N_2(w)$. Let $x\in N_S(C_j)\backslash N_S(C_i)$. By Claim 2 and the hypothesis of this theorem, x is adjacent to a vertex y in $N_S(C_i)$, otherwise $(N_S(C_i)\backslash\{w\})\cup\{x\}$ is an independent set of order k in $N_2(w)$, a contradiction. So G has a 5-cycle xC_jwC_iyx. Note that w exists by Claim 2. Since $N(x)\cap N(y)=\emptyset$ for $g\geqslant 4$, either $|N(x)\cap\{C_1,C_2,\ldots,C_m\}|\leqslant m/2$ or $|N(y)\cap\{C_1,C_2,\ldots,C_m\}|\leqslant m/2$. Without loss of generality, assume the former is the case. Then $|N_S(x)|\geqslant k-m/2$. But $N_S(x)\cap N_S(C_j)=\emptyset$ for $g\geqslant 4$. So $v(G)\geqslant |N_S(C_j)|+|N_S(x)|+|\{C_1,C_2,\ldots,C_m\}|\geqslant k+(k-m/2)+m=2k+m/2$. However, $m=|S'|+r$ $(r\geqslant 2)$ and $v=2\lceil k/4\rceil+|S'|+m=2\lceil k/4\rceil+m-r+m\geqslant 2k+m/2$. So $m\geqslant 2(2k-2\lceil k/4\rceil+r)/3\geqslant (4k-4\lceil k/4\rceil+4)/3=k+(k+4-4\lceil k/4\rceil)/3>k$, contradicting Claim 3. This last contradiction completes the proof of Theorem 1. $\square$

Corollary 2. *Let G be a graph with even order. If $\alpha(G)\leqslant\kappa(G)$ and $g(G)\geqslant 4$, then G is regular and G is $\lceil k/4\rceil$-extendable, where $k=d(v)$ for $v\in V(G)$.*

Proof. Suppose $\alpha(G)\leqslant\kappa(G)$ and $v\in V(G)$. If $\alpha(G[N_2(v)])\geqslant d(v)$, then a maximum independent set S in $N_2(v)$ and v form an independent set with $|S\cup\{v\}|\geqslant d(v)+1\geqslant\kappa+1$, contradicting the hypothesis that $\alpha(G)\leqslant\kappa(G)$. Hence a graph satisfying the hypotheses of Corollary 2 also satisfies those of Theorem 1. The result follows. $\square$

Remark 1. In the following, we construct a family of graphs to show that, under the hypotheses of Theorem 1 and Corollary 2, some graphs are not $\lfloor k/2\rfloor$-extendable, where $k=d(v)$, and when $k=4t$ $(t\geqslant 1)$, the graphs are not $(\lceil k/4\rceil+1)$-extendable. Hence Theorem 1 and Corollary 2 are sharp.

Let G be a graph with $V(G)=S\cup T$. Here $S\cap T=\emptyset$, $|T|=m$ and $|S|=k+m-r$, where r is a positive even number and $2\leqslant m\leqslant k$. Let $T=T_1\cup T_2$ such that $T_1\cap T_2=\emptyset$, $|T_1|=k-m+r$ and $|T_2|=2m-k-r$. And let $S=S_1\cup S_2\cup S_3\cup S_4$ such that $S_i\cap S_j=\emptyset$

$(1 \leqslant i < j \leqslant 4)$, $|S_1| = k/2, |S_2| = k/2$, $|S_3| = k - m$ and $|S_4| = 2m - k - r$. Let $E(G) = \{uv \mid u \in S_1 \text{ and } v \in T\} \cup \{uv \mid u \in S_2 \text{ and } v \in T_1\} \cup \{uv \mid u \in S_4 \text{ and } v \in T_2\} \cup \{uv \mid u \in S_1 \text{ and } v \in S_3\} \cup \{uv \mid u \in S_2 \text{ and } v \in S_3 \cup S_4\}$. When $2m - k - r = k/2$, i.e. when $m = (3k + 2r)/4$, G is a graph satisfying the hypotheses of Theorem 1 and Corollary 2, which is not $\lfloor k/2 \rfloor$-extendable as $G[S]$ contains $\lfloor k/2 \rfloor$ independent edges but $o(G-S) = |T| = |S| - k + r$ $(r \geqslant 2)$. Furthermore, when $k = 4t$ and $r = 2t$ $(t \geqslant 1)$, G is not $(\lceil k/4 \rceil + 1)$-extendable as $o(G - S) = |S| - k + r = |S| - 4t + 2t > |S| - 2t - 2$.

Remark 2. Although it is not easy to construct graphs satisfying the hypotheses of Theorem 1 and Corollary 2, Bauer et al. [2] showed that there is a large variety of such graphs by giving a clever construction method.

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